ISRAEL JOURNAL OF MATHEMATICS 154 (2006), 51–59

COVERS OF VARIETIES WITH COMPLETELY SPLIT POINTS

BY

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ABSTRACT

Let K be a field such that all Sylow subgroups of its absolute Galois group G_K are infinite. Let X be a smooth variety over K with function field F and $Y \to X$ the normalisation in a finite, separable extension $E|F$. We show: If there is a closed point $x \in X$ which does not split completely in $Y \to X$, then the set of these points is Zariski dense in X.

1. Introduction

Let C be a curve over a field K. In order to develop the class field theory of C, the covers of C need to be examined locally and this information collected into global information (e.g. see [6] for a p-adic field K). This approach is limited to covers of C with nontrivial local behaviour. The covers of C which cannot be treated in this manner are those in which all closed points split completely. The function field extension associated with such a cover is called a c.s. extension.

The compositum of two finite c.s. extensions of a function field F is again a c.s. extension. Thus one is led to consider the compositum $F_{\text{c.s.}}$ of all finite c.s. extensions of F. In [6, Theorem 7.1 (2)], Saito describes the group $\text{Gal}(F_{\text{c.s.}}|F)$ in the case that F is a function field of one variable over a local field K. He shows that it is isomorphic to the profinite completion of the fundamental group of the dual graph of a suitable reduction of the non-singular projective model X of F.

An important step in Saito's proof is [6, Theorem 7.1 (1)]. We prove the following generalisation of this result: Let K be a field such that all Sylow

Received April 19, 2005 and in revised form May 18, 2005

subgroups of its absolute Galois group G_K are infinite. Let X be a smooth, irreducible variety over K with function field F. Let $Y \to X$ be the normalisation of X in a finite, separable extension $E|F$. If there is a closed point $x \in X$ which is not completely split in $Y \to X$, then the set of these points is Zariski dense in X. For K a local field and $\dim X = 1$ this is Saito's result [6, Theorem 7.1 (1)].

Our proof of the general case is more elementary than Saito's proof which uses the arithmetic of local fields, in particular the class field theory of complete two dimensional local rings.

The hypothesis that all Sylow groups are infinite means that K is "not too large" in a certain sense. For example, all Hilbertian fields have this property. Note that for Hilbertian fields the existence of non-split points is guaranteed (see Remark 3). Therefore our theorem can be viewed as a generalisation of properties of Hilbertian fields.

We define a c.s. cover to be a finite cover $Y \to X$ of a smooth variety X over K such that every closed point of X splits completely in Y. Our main theorem implies that this is a birational property, i.e., depends only on the corresponding function field extension, provided that all Sylow subgroups of G_K are infinite.

Notation: K is a field with absolute Galois group G_K. For most of the paper we assume that all Sylow subgroups of G_K are infinite.

2. The main result

A **variety** over a field K is an integral, separated scheme of finite type over $\operatorname{Spec} K$. A **function field** over K is a finitely generated, separable extension of K. Unless stated otherwise, all varieties and function fields will be over K.

Let $Y \to X$ be a finite cover of normal varieties. Then a closed point $x \in X$ is **completely split** in $Y \to X$ if the number of closed points of Y over x equals the degree of $Y \to X$.

THEOREM 1: *Let K be a field such that all Sylow subgroups of G_K are infinite. Let X be a smooth variety over K with function field F. Let $E|F$ be a finite separable extension and Y the normalisation of X in $E|F$. If there is a closed point of X which does not split completely in $Y \to X$ then the set of these points is Zariski dense in X.*

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COROLLARY 2: *Let K be a field such that all Sylow subgroups of G_K are infinite. Let $E|F$ be a finite separable extension of function fields over K. Then the following are equivalent:*

1. *There exists a smooth variety X over K with function field F such that every closed point of X splits completely in the normalisation of X in E.*
2. *The same holds for all smooth varieties X over K with function field F.*

If the equivalent conditions (1) and (2) hold, we say $E|F$ is a **c.s. extension** (completely split extension).

Proof (of the corollary): Let $X' \subseteq X$ be a non-empty open subvariety. If all closed points of X' split completely in Y then the same holds for X by Theorem 1. The claim follows because any two varieties with the same function field have isomorphic open subvarieties. ∎

Lemma 5 (5) implies that the compositum of two c.s. extensions of F is again a c.s. extension. Thus we are led to consider the compositum $F_{\text{c.s.}}|F$ of all (finite) c.s. extensions of F, which we call the **maximal c.s. extension** of F. This generalises the definition of Saito [6, Theorem 7.1] in the case that F is a function field of one variable over a local field. In this case, $\mathrm{Gal}(F_{\text{c.s.}}|F) = \widehat{\pi_1(\Gamma)}$ is the profinite completion of the fundamental group of the dual graph of a suitable reduction of the non-singular projective model X of F.

Remark 3: Assume that all Sylow subgroups of G_K are infinite, and K is either Hilbertian, PAC or finite. Then $F_{\text{c.s.}} = F$.

Proof: Let $E|F$ and $Y \to X$ as in Theorem 1 with $E \neq F$. We have to show there is a closed point of X which does not split completely in Y. By Lemma 5, (1), (3), we may assume $E|F$ is Galois of prime degree p. Let $G = \mathrm{Gal}(E|F)$.

Let K be Hilbertian. Choose a separating transcendency base $t_1, \dots, t_d$ for $F|K$. The finite separable field extension $F|K(t_1, \dots, t_d)$ induces a rational map $X \to \mathbb{P}^d_K$. Consider the composition $f\colon Y \to X \to \mathbb{P}^d_K$. By the Hilbert property of K there is a K-rational point $z \in \mathbb{P}^d_K$ where f is defined and such that z has only one point of Y over it. Then the point x of X over z has only one pre-image in Y.

Let K be PAC such that the absolute Galois group of K has infinite Sylow subgroups. We may assume K is the field of constants of F. Let L be the field of constants of E. If $L \neq K$ then all K-rational points of X do not split completely in Y. Now assume $L = K$. There are algebraic extensions $K''|K'|K$ such that $K''|K'$ is Galois of degree p. We have $\mathrm{Gal}(EK'|FK') = \mathrm{Gal}(E|F)$. Therefore by Lemma 5 (4), we may assume $K' = K$. By the field crossing argument (see [2, proof of Lemma 24.1.1] or [3, Lemma 1]) there is a K-rational point of X with residue field extension $K''|K$ in $Y \to X$.

If K is finite, let $L|K$ be the Galois extension with group $\prod_{l \neq p} \mathbb{Z}/l$. Then $EL \neq FL$. By Lemma 5 (4) we may replace K by L. But L is PAC by [2, Corollary 11.2.4] and its absolute Galois group has infinite Sylow groups. ∎

Remark 4: (a) Note that even in the PAC case, the point of X that doesn't split completely in Y need not be K-rational: If K has no Galois extension of degree p and $Y \to X$ is Galois of degree p, then any K-rational point of X splits completely in Y.

(b) Let K be perfect, not PAC, and $G_K \cong \mathbb{Z}_p$ (then K is ample by [4]). Let $L|K$ be the unique Galois extension of degree p. Let X be a smooth absolutely irreducible K-variety without a K-rational point. Then the residue field of every closed point of X contains L. Let $X_L = X \times_K L$. Then all closed points of X split completely in $X_L \to X$. Hence $F_{\text{c.s.}} \neq F$.

(c) Here we show that the assumption made on G_K in Theorem 1 that all Sylow subgroups are infinite is necessary. Note first that a p-Sylow subgroup of G_K is infinite, iff it is nontrivial and not of order 2 if $p = 2$.

Let $Y \to X$ be a ramified Galois cover of degree p. Clearly the ramified points do not split completely. Conversely, an unramified point splits completely, if its residue field has no extension of degree p.

Therefore it suffices to construct X such that none of the residue fields of the closed points has an extension of degree p. If the p-Sylow subgroup of G_K is trivial, this is the case for all X.

If $p = 2$ and the 2-Sylow subgroup of G_K has order 2, then K is formally real. Let X be the curve with equation $u^2 + v^2 + 1 = 0$. None of the residue fields is formally real. Therefore, none of them has an extension of degree 2.

3. The proof of Theorem 1

LEMMA 5: *Let X be a smooth variety over K with function field F. Let $E|F$ be a finite separable extension of degree n and Y the normalization of X in $E|F$. Let $x \in X$ be a closed point.*

1. *Let $E'|E$ be a finite separable extension and Y' the normalization of Y in E'. Then x splits completely in $Y' \to X$, if and only if x splits completely in $Y \to X$ and every pre-image of x in Y splits completely in $Y' \to Y$.*
2. *Suppose $E|F$ is Galois with group G. Then x splits completely in Y if and only if the decomposition group G_y is trivial for any closed point y of Y over x.*
3. *Let $\hat{E}$ be the Galois closure of $E|F$ and $\hat{Y}$ the normalisation of X in $\hat{E}$. Then x splits completely in $Y \to X$ if and only if it splits completely in $\hat{Y} \to X$.*
4. *Let $F'|F$ be a finite extension and $E' = EF'$ the compositum (in some algebraic closure of F). Let X' and Y' be the normalisations of X in F' and E'. If x splits completely in $Y \to X$ then every pre-image of x in X' splits completely in $Y' \to X'$.*
5. *Assume in (4) additionally that $F'|F$ is separable. Then x splits completely in Y' if and only if it splits completely in Y and X'.*

Proof:

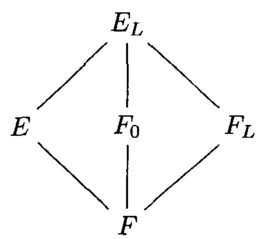
1. Let $m = \deg(Y'|Y)$. Then $\deg(Y'|X) = mn$. The point x has mn pre-images on Y' iff x has n pre-images on Y and each pre-image of x on Y has m pre-images on Y'.
2. It is well-known that G acts transitively on the closed points y of Y over x. Therefore, the number of these points is $n = |G|$ if and only if all decomposition groups are trivial.
3. Let $G = \mathrm{Gal}(\hat{E}|F)$ and $H = \mathrm{Gal}(\hat{E}|E)$. Let $\hat{y}$ be a point of $\hat{Y}$ over x. The points of $\hat{Y}$ (resp., Y) over x correspond to the coset space $G/G_{\hat{y}}$ (resp., the orbits of H on this coset space). The orbits of H on $G/G_{\hat{y}}$ correspond to the double coset space $H\backslash G/G_{\hat{y}}$.
   Assume now x splits completely in Y, i.e. $|H\backslash G/G_{\hat{y}}| = n = [G : H]$. This implies $G_{\hat{y}} \leq H$. Then $G_{\hat{y}}$ is also contained in all conjugates of H in G, hence in their intersection. This intersection is trivial because $\hat{E}$ is the Galois closure of $E|F$. Therefore $G_{\hat{y}} = 1$, i.e., x splits completely in $\hat{Y}$. The converse follows from (1).
4. By (3) we may assume $E|F$ is Galois. Then also $E'|F'$ is Galois and $G' := \mathrm{Gal}(E'|F')$ embeds into $G := \mathrm{Gal}(E|F)$ via restriction to E. Under this embedding, the decomposition group in G' of a point of Y' embeds into the decomposition group of its image in Y. By (2) this implies the claim.
5. This follows from (1) and (4). ∎

Proof of Theorem 1: In this proof "point" means closed point of a variety. The proof of Theorem 1 consists of a series of reduction steps to the following special cases:

- K is the exact field of constants of X.
- K is perfect: If $p = \mathrm{char}(K)$, let $K_{\mathrm{insep}} := \bigcup_n K^{1/p^n}$ be the maximal algebraic inseparable extension of K. K_{insep} is perfect. Define X' resp. Y' as the normalisation of X resp. Y in FK_{insep} resp. EK_{insep}. The maps $X' \to X$ and $Y' \to Y$ are radicial and hence induce bijections on the sets of closed points (cf. [5, II Remark 3.17]). A point $x' \in X'$ is completely split in $Y'|X'$, iff its image x on X is completely split in $Y \to X$. Now replace X by X' and K by K_{insep}. Note that for perfect K a variety is smooth iff it is regular.
- Let F_0 be an intermediate field of $E|F$ and X_0 the normalisation of X in F_0. Then we may assume that all these finitely many X_0 are regular:
  Delete from X the images of the singular loci of all the X_0. We have to show that there still is a point of X that does not split completely in Y. This is guaranteed if the original point x was unramified in $Y \to X$, because then its pre-images in X_0 were regular.
  On the other hand, if x was ramified, then we may replace x by any other ramified point. The purity of the branch locus ([1, X.3.1]) says that the branch locus $\mathrm{Ram}(Y \to X)$ has pure codimension 1 in X. But as X_0 is normal the singular locus of X_0 has codimension 2 in X_0 and therefore the cover $Y \to X$ is still ramified after deleting the singular loci.
- $E|F$ is Galois: Let $\hat{E}|F$ be the Galois closure of $E|F$. By Lemma 5 (3) we may replace $E|F$ by $\hat{E}|F$.
- $E|F$ has prime degree p: Let y be a point of Y over x. Let G_y be the (nontrivial) decomposition group of y in $\mathrm{Gal}(E|F)$. There is a subgroup $G_0 \leq G_y$ of prime order p. Let F_0 be the fixed field of G_0. The extension $E|F_0$ is Galois of degree p. Let X_0 be the normalisation of X in F_0. The construction of X_0 implies that the image of y in X_0 is not completely split in $Y \to X_0$. If a point in X_0 does not completely split in $Y \to X_0$, then its image in X does not completely split in $Y \to X$. Now replace X by X_0.
- The point x is a K-rational point on X: The residue field $L = \kappa(x)$ of x is a finite extension of K. This is separable, because K is perfect. Let $Y_L \to X_L$ be the cover obtained by base change from K to L. There is a point x_L over x in X_L, which has residue field L. This point x_L does not

split completely in $Y_L \to X_L$. The variety X_L is regular, because $X_L \to X$ is étale. If a point in X_L does not split completely in $Y_L \to X_L$ then its image in X does not split completely in $Y \to X$. Now replace $Y \to X$ by $Y_L \to X_L$.

- The field K has a Galois extension $L|K$ of degree p: If x is unramified in $Y \to X$ then the residue extension gives such an L. If x is ramified in $Y \to X$, we may replace $E \to F$ by $EM|FM$ where M is a finite separable extension of K (by Lemma 5 (4)). The assumption that the p-Sylow subgroup of G_K is infinite guarantees that there is such an M, which has a Galois extension of degree p.
- $E \cong F \otimes_K L$: If this is not the case, then consider the composite field $E_L = EL$. Let $F_L = FL$. The extension $E_L|F$ is Galois with group $G = \mathbb{Z}/p \times \mathbb{Z}/p$. Let $F_0 \neq E, F_L$ be one of the $p+1$ proper intermediate fields and X_0 (resp., Y_L) the normalisation of X in F_0 (resp., in E_L). Then $E_L \cong F_0 \otimes_K L$. In particular, this implies that X_0 is regular, because Y_L is regular.



If the points of X_0 which do not split completely in Y_L are dense in X_0, then the images of these points are dense in X and do not split completely in Y by Lemma 5 (4). It remains to show that there is a point $x_0 \in X_0$ that does not split completely in Y_L. Then we can replace $Y \to X$ by $Y_L \to X_0$, which satisfies the desired property $E_L \cong F_0 \otimes_K L$.

Since G is abelian, all points of Y_L over x have the same decomposition group $H \leq G$. The group H cannot be contained in $\mathrm{Gal}(E_L|E)$ (resp., in $\mathrm{Gal}(E_L|F_L)$), otherwise x would split completely in $E|F$ (resp., $F_L|F$); recall that x is a K-rational point of X, therefore it does not split completely in $X_L = X \times_K L$. Therefore we may choose F_0 such that $H \geq \mathrm{Gal}(E_L|F_0)$. Then each pre-image x_0 of x in X_0 does not split completely in $Y_L \to X_0$, as desired.

- X is a curve: Since x is regular on X there is a system of parameters $a_1, \ldots, a_d$ for the regular local ring $A = \mathcal{O}_{X,x}$. The equations $a_2 = \ldots = a_n = 0$ define a regular curve C in an affine neighbourhood of x in X.

The parameters can be chosen such that C meets any given open subset of X (since X is irreducible).

As $Y \to X$ is étale the pre-image D of C in Y is étale over C. Furthermore x has only one pre-image in D, since $\deg(Y \to X) = p$. Therefore D is irreducible. It is the normalisation of C in the function field of D. Now replace $Y \to X$ by $D \to C$.

Conclusion of the proof: We have reduced to the case that E and F are function fields of one variable and $E = F \otimes_K L$, where $L|K$ is Galois of prime degree p. Thus a point $x' \in X$ splits completely in $Y \to X$, iff its residue field $\kappa(x')$ contains L. Then the point has a degree which is divisible by p. It remains to show that there are infinitely many $x' \in X$ such that $\kappa(x')$ does not contain L. This follows from the lemma below, because the given point x of X is K-rational.

LEMMA 6: *Let C be a regular curve over a field K. If there is a point x of C whose degree is not divisible by a prime p, then there are infinitely many.*

Proof: The proof is the same as for Proposition 1 in [4]. For the convenience of the reader, we reproduce it here. Recall that the degree of a point $x \in C$ is $\deg(\kappa(x)|K)$.

We may assume C is complete. Let $x_1, \dots, x_m$ be additional points of C. Use the weak approximation theorem to choose f in the function field of C which has a simple zero at x and value 0 at the points $x_1, \dots, x_m$. Then $\mathrm{div}(f) = x + \sum_{j=1}^{n} k_j y_j$ for some additional distinct points $y_1, \dots, y_n$. It follows that

$$0 = \deg(\mathrm{div}(f)) = \deg(x) + \sum_{j=1}^{n} k_j \deg(y_j).$$

As $\deg(x)$ is not divisible by p, there must be one of the y_j whose degree is not divisible by p. So C has infinitely many points of degree prime to p. ∎

ACKNOWLEDGEMENT: I thank M. Jarden and H. Völklein for helpful remarks on this paper.

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